Electrostatic Analysis for Plane Problems With Finite Formulation

Paolo Bettini and Francesco Trevisan

***Abstract*—The *Finite Formulation* is an alternative to the widely used Finite Elements or Edge Elements methods and adopts the global variables as a starting point. The Finite Formulation allows one to straightforwardly deduce the set of algebraic equations. This method has been applied to the solution of plane electrostatic problems on the basis of two different topologies of the primal mesh elements: triangular and quadrilateral. The obtained results in the two cases have been compared each other and also with those from the Finite Elements.**

***Index Terms*—Edge elements, electrostatic, finite formulation.**

I. INTRODUCTION

IN ORDER to solve the field problems with a numerical method, an algebraic formulation is needed, usually obtained through a discretization process of the original differential or integral equations. Therefore, different discretization processes have been developed and applied with success in computational electromagnetism, such as: the Finite Difference or Finite Differences in Time Domain, the Finite Element method [2], [3], the Edge Element method [4], and the Finite Integration Theory [1], [5], [6].

As an alternative, it is possible to reformulate field laws in finite form so that an algebraic system of equations can be directly written to solve the field problem, avoiding the use of a discretization process applied to a differential equation. This approach is the Finite Formulation [7] and the corresponding numerical method is known as the Cell method. The Cell method has been applied, with promising results, to the numerical solution of electromagnetic field problems [8], [11], [13] and in other fields of computational physics such as the fluid acoustic [9], the elastodynamic, and the heat conduction [10]. It is interesting, but not surprising, to see that using this approach, completely independent from the differential formulation, the same results of the Finite Integration theory or of the Finite Elements can be derived. The fact that the same result can be obtained from completely different starting points is typical of science. In particular, the Cell method in electromagnetism can be viewed as an extension and a generalization of the Finite Integration theory, being based on a mathematically more rigorous description provided by the use of algebraic topology. Anyway, these two formulations remain different in their conceptual core because the respective starting hypotheses are different. The aim of this paper is to apply the Cell method to the two-dimensional (2-D) electrostatic problem of the computation of the capacitance of a transmission line in the presence of nonhomogeneous media.

TABLE I
GLOBAL VARIABLES OF INTEREST IN ELECTROSTATIC AND THEIR RELATION WITH FIELD VARIABLES

Configuration global variables (V)	*Source global variables* (C)
electric voltage: $U[\mathbf{L}] = \int_{\mathbf{L}} \mathbf{E} \cdot d\mathbf{L}$	*electric flux*: $\Psi[\tilde{\mathbf{S}}] = \int_{\tilde{\mathbf{S}}} \mathbf{D} \cdot d\mathbf{S}$
electric potential: $V[\mathbf{P}]$	*electric charge content:* $Q[\tilde{\mathbf{V}}] = \int_{\tilde{\mathbf{V}}} \rho dV$

II. FINITE FORMULATION FOR ELECTROSTATIC

The starting point of the Finite Formulation is the use of *global variables*. The global variables refer to oriented geometrical elements of a system like points $\mathbf{P}$, lines $\mathbf{L}$, surfaces $\mathbf{S}$, volumes $\mathbf{V}$, instants $\mathbf{I}$, and intervals $\mathbf{T}$. Global variables are continuous in the presence of different materials and do not require any restriction, like field functions, in terms of derivability conditions on the material media parameters.

The global variables relevant to our electrostatic problem are reported in Table I. Their dependence on the oriented geometrical elements (in bold face) is evidenced within a square bracket; moreover, to distinguish one of the two possible orientations (*inner* and *outer*), a tilde is used to specify the outer orientation with respect to the inner orientation (without tilde). The global variables are related to the field functions by means of an integration performed on oriented lines, surfaces, volumes, and time intervals; therefore, they are equivalent to the commonly used integral variables.

According to Finite Formulation, global variables can be also classified into *configuration, source,* and *energy* variables. The *configuration variables* describe the configuration of the field with the intervention of the material parameters. The *source variables* describe the sources of the field without involving the material parameters. The energy variables are the product between a configuration and a source variable.

A. Cell Complexes

The classification above has a great impact in the numerical applications of the Finite Formulation. It is matter of fact that the Cell method requires the use of a pair of oriented cell complexes $K = \{\mathbf{p}_h, \mathbf{l}_i, \mathbf{s}_j, \mathbf{v}_k\}$, and $\tilde{K} = \{\tilde{\mathbf{v}}_h, \tilde{\mathbf{s}}_i, \tilde{\mathbf{l}}_j, \tilde{\mathbf{p}}_k\}$ one dual of the other, endowed with inner and outer orientation, respectively; see Fig. 1. Two complexes are needed because the configuration variables refer to a cell complex, the primal, and the source

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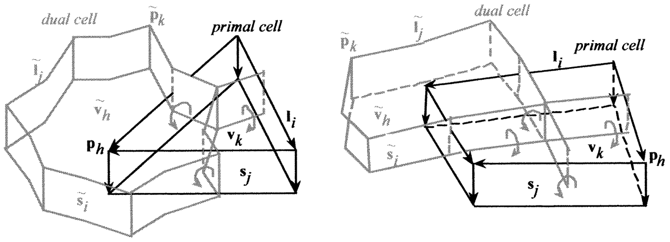

Fig. 1. Primal cell complex based on triangles (on the left) and primal cell complex based on quadrilateral elements (on the right). The corresponding dual complexes are also shown staggered with respect to the primal ones.

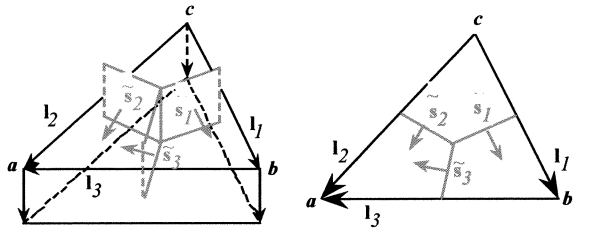

Fig. 2. Primal cell, based on triangles, and portions of dual faces; local numbering is adopted.

variables are referred to the dual complex. Therefore, the association of the global variables to the respective primal or dual cell complex, both in space and in time, becomes unique.

Two kinds of topologies of the primal cell complex K have been considered here for the 2-D electrostatic problem, based on a set of prisms $\mathbf{v}_k$ with triangular or quadrilateral base and unit depth, respectively; see Fig. 1. The corresponding *dual cell* complexes $\tilde{K}$ are derived according to the barycentric subdivision: as dual points $\tilde{\mathbf{p}}_k$ the barycentres of the prisms are considered while the dual edges $\tilde{\mathbf{l}}_j$ are broken lines linking the barycentres of two adjacent cells $\mathbf{v}_k$ and passing via the barycentre of the common face $\mathbf{s}_j$. For the primal complex K, being the mesh of two dimensions, the number of points is the number of mesh nodes N, the number of edges is the number of mesh edges L, the number of faces is L, and the number of cells is the number of mesh elements E. For the dual complex $\tilde{K}$ the number of points is E, the number of edges is L, the number of faces is L, and the number of cells is N.

B. Physical Laws for Electrostatic

The *physical laws* of electromagnetism link physical variables of the same kind (configuration variables with configuration variables and source variables with source variables) and express that a global variable associated to a geometrical element is equal to another global variable associated to its boundary. In the case of electrostatic, with reference to a pair of primal dual cell complexes K-$\tilde{K}$ above defined, the laws are

$$\begin{aligned} \tilde{\mathbf{D}}\Psi &= \mathbf{Q} \text{ (Gauss law)} && (1)\\ \mathbf{C}\mathbf{U} &= 0 \text{ (voltage is conservative)} && (2) \end{aligned}$$

where $\tilde{\mathbf{D}}$ is the $N \times L$ matrix of incidence numbers between the outer orientations of volumes $\tilde{\mathbf{v}}_h$ and faces $\tilde{\mathbf{s}}_i$ of the dual complex; $\mathbf{C}$ is the $E \times L$ matrix of incidence numbers between the inner orientations of faces $\mathbf{s}_j$ and lines $\mathbf{l}_i$ of the primal complex. $\mathbf{U}$ is the vector of electric voltages associated to primal edges $\mathbf{l}_i$ and Ψ is the vector of electric fluxes associated to dual faces $\tilde{\mathbf{s}}_i$; $\mathbf{Q}$ is the vector of the electric charge content associated to dual cells $\tilde{\mathbf{v}}_h$. From the identity $C\ G = 0$, where $\mathbf{G}$ is the $L \times N$ matrix of incidence numbers between the inner orientations of lines $\mathbf{l}_i$ and points $\mathbf{p}_h$ of the primal complex, the vector $\mathbf{V}$ of electric potentials associated to primal nodes $\mathbf{p}_h$ can be introduced such that

$$\mathbf{U} = -\mathbf{G}\mathbf{V}. \tag{3}$$

In this way, (2) becomes identically satisfied. From the duality between orientations of the cell complexes K-$\tilde{K}$, the following relation holds:

$$-\mathbf{G}^T = \tilde{\mathbf{D}}. \tag{4}$$

III. Electric Constitutive Equation

The link between configuration and source variables is given by the *constitutive equations* that contain material properties and metrical notions. Two approaches will be described here in deducing the electric constitutive equation.

1) A first approach assumes a uniform $\mathbf{E}$ and $\mathbf{D}$ fields inside each primal cell $\mathbf{v}_k$ with triangular base.
2) A second approach, more general, assumes the uniformity of the fields in subregions of each primal cell $\mathbf{v}_k$ [11]; it is applied to the case of primal cells with quadrilateral base.

A. Case of Triangular Elements

Considering a primal cell with triangular base and local numbering of nodes and edges as shown in Fig. 2, the uniform electric field vector $\mathbf{E}^e$ can be equivalently deduced as

$$\mathbf{E}^e = (\mathbf{L}_a)^{-1}\mathbf{U}_a, \quad \mathbf{E}^e = (\mathbf{L}_b)^{-1}\mathbf{U}_b \tag{5}$$

where $\mathbf{L}_a$, $\mathbf{L}_b$ are 2×2 matrices whose rows are the row vectors of a pair of primal edges $\mathbf{l}_2$, $\mathbf{l}_3$ and $\mathbf{l}_1$, $\mathbf{l}_3$, respectively; $\mathbf{U}_a = [U_2, U_3]^T$, $\mathbf{U}_b = [U_1, U_3]^T$ are the vectors of the corresponding electric voltages. Introducing the element voltage vector $\mathbf{U}_e = [U_1 \quad U_2 \quad U_3]^T$, (5) can be rewritten as

$$\mathbf{E}^e = \frac{1}{2}(\mathbf{A} + \mathbf{B})\mathbf{U}^e \tag{6}$$

where $\mathbf{A}$, $\mathbf{B}$ are two 2×3 matrices whose columns are those of $(\mathbf{L}_a)^{-1}$, $(\mathbf{L}_b)^{-1}$ respectively; in addition, $\mathbf{A}$ has the first column of zeros and $\mathbf{B}$ has the second column of zeros.

On the other hand, the vector of electric fluxes $\Psi^e = [\Psi_1 \quad \Psi_2 \quad \Psi_3]^T$ relative to the portions of dual faces $\tilde{\mathbf{s}}_1$, $\tilde{\mathbf{s}}_2$, $\tilde{\mathbf{s}}_3$ tailored inside the primal cell $\mathbf{v}_k$ can be written as

$$\Psi^e = \tilde{\mathbf{S}}^e\mathbf{D}^e \tag{7}$$

where $\tilde{\mathbf{S}}^e$ is the 3×2 matrix which rows are the portions of dual area vectors $\tilde{\mathbf{s}}_1$, $\tilde{\mathbf{s}}_2$, $\tilde{\mathbf{s}}_3$. From the constitutive equation between the uniform fields, $\mathbf{D}^e = \varepsilon^e\mathbf{E}^e$ with ε^e the 2×2 permittivity

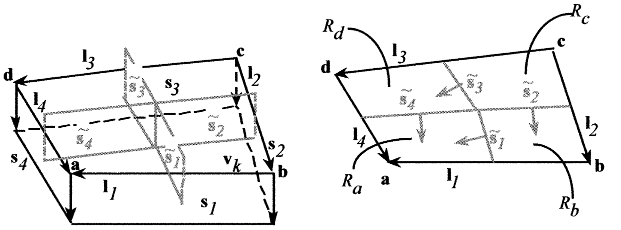

Fig. 3. Primal cell, based on quadrilateral elements and portions of dual faces; local numbering is adopted. The four subregions are evidenced on the right.

matrix, the final elemental constitutive equation can be deduced from (6) and (7) as

$$\Psi^e = \frac{1}{2}\tilde{\mathbf{S}}^e \varepsilon^e (\mathbf{A}+\mathbf{B})\mathbf{U}^e = \mathbf{M}^e_\varepsilon \mathbf{U}^e \tag{8}$$

where $\mathbf{M}^e_\varepsilon$ is the 3×3 nonsymmetric elemental matrix.

B. Case of Quadrilateral Elements

Inside each primal cell four subregions R_a, R_b, R_c, and R_d can be defined, delimited by portions of dual edges and portions of primal edges; see Fig. 3. Within each subregion the fields $\mathbf{D}_i$, $\mathbf{E}_i$ with $i = a$, *b*, *c*, *d*, are assumed uniform. Considering first the subregion R_a, from the vector of electric fluxes $\Psi_a = [\Psi_1 \quad \Psi_4]^T$, relative to the portions of dual faces $\tilde{\mathbf{s}}_1$, $\tilde{\mathbf{s}}_4$, the field $\mathbf{D}_a$ can be deduced as: $\mathbf{D}_a = (\tilde{\mathbf{S}}_a)^{-1}\Psi_a$ being $\tilde{\mathbf{S}}_a$, the 2×2 matrix whose rows are the row vectors of the portions of dual areas $\tilde{\mathbf{s}}_1$, $\tilde{\mathbf{s}}_4$. For a generic subregion R_i, with $i = a, b, c, d$, and from the corresponding vector of electric fluxes Ψ_i, $\mathbf{D}_i$ can be derived as

$$\mathbf{D}_i = \left(\tilde{\mathbf{S}}_i\right)^{-1} \Psi_i, \text{ with } i = a, b, c, d \tag{9}$$

where $\tilde{\mathbf{S}}_i$ is the 2×2 matrix whose rows are the row vectors of the portions of dual areas delimiting the region R_i. Introducing the element vector of electric fluxes $\Psi^e = [\Psi_1 \quad \Psi_2 \quad \Psi_3 \quad \Psi_4]^T$, relative to the four portions of dual areas $\tilde{\mathbf{s}}_1, \ldots \tilde{\mathbf{s}}_4$, (9) can be rewritten as

$$\mathbf{D}_i = \mathbf{A}_i \Psi^e \text{ with } i = a, b, c, d \tag{10}$$

where $\mathbf{A}_i$ is the 2×4 matrix whose columns are those of $(\tilde{\mathbf{S}}_i)^{-1}$, and, in addition, it has two columns of zeros corresponding to the two electric fluxes not belonging to the region R_i.

On the other hand, the voltages relative to primal edges can be expressed as

$$\begin{aligned} U_1 &= \frac{1}{2}\mathbf{l}_1\mathbf{E}_a + \frac{1}{2}\mathbf{l}_1\mathbf{E}_b, \quad U_2 = \frac{1}{2}\mathbf{l}_2\mathbf{E}_b + \frac{1}{2}\mathbf{l}_2\mathbf{E}_c \\ U_3 &= \frac{1}{2}\mathbf{l}_3\mathbf{E}_c + \frac{1}{2}\mathbf{l}_3\mathbf{E}_d, \quad U_4 = \frac{1}{2}\mathbf{l}_4\mathbf{E}_d + \frac{1}{2}\mathbf{l}_4\mathbf{E}_a \end{aligned} \tag{11}$$

where the $\mathbf{l}_i$ with $i = 1, \ldots, 4$ are the row vectors of the four primal edges. Introducing the elemental vector of voltages $\mathbf{U}^e = [U_1 \quad U_2 \quad U_3 \quad U_4]^T$, (11) can be rewritten as

$$\mathbf{U}^e = \frac{1}{2}\mathbf{L}_a\mathbf{E}_a + \frac{1}{2}\mathbf{L}_b\mathbf{E}_b + \frac{1}{2}\mathbf{L}_c\mathbf{E}_c + \frac{1}{2}\mathbf{L}_d\mathbf{E}_d \tag{12}$$

where $\mathbf{L}_i$, with $i = a, b, c, d$, is the 4×2 matrix whose nonzero rows are those corresponding to the couple of primal edge row vectors delimiting the subregion R_i.

From the constitutive equation between the uniform fields, $\mathbf{E}_i = \varepsilon^{e-1}\mathbf{D}_i$, the elemental constitutive equation can be deduced from (10) and (12) as

$$\mathbf{U}^e = \frac{1}{2}\left(\sum_{i=a,b,c,d} \mathbf{L}_i \varepsilon^{e-1} \mathbf{A}_i\right)\Psi^e. \tag{13}$$

By inversion of the 4×4 matrix in (13), the elemental matrix $\mathbf{M}^e_\varepsilon$ can be computed such that

$$\Psi^e = \mathbf{M}^e_\varepsilon \mathbf{U}^e. \tag{14}$$

It can be shown that in the particular case of triangular elements, as in Fig. 2, this approach leads to exactly the same elemental matrix $\mathbf{M}^e_\varepsilon$ of (8).

Both for the case of triangular elements and for case of the quadrilateral elements, starting from the local constitutive (8) or (14), the global constitutive equation can be assembled working element by element as

$$\Psi = \mathbf{M}_\varepsilon \mathbf{U} \tag{15}$$

where $\mathbf{M}_\varepsilon$ is the $L \times L$ nonsymmetric constitutive matrix while Ψ, $\mathbf{U}$ are the global vectors of electric fluxes and electric voltages, respectively.

IV. NUMERICAL RESULTS AND DISCUSSION

The algebraic system to be solved can be derived by substituting in (1) the global constitutive (15) where $\mathbf{U}$ is expressed by means of (3), obtaining

$$-\left(\mathbf{G}^T \mathbf{M}_\varepsilon \mathbf{G}\right)\mathbf{V} = \mathbf{Q}. \tag{16}$$

It can be shown that, in the case of triangular elements under the hypothesis of uniform field and using a dual mesh with barycentric subdivision, the resulting system matrix in (16) is symmetric [12]. Moreover, this matrix is coincident with the final system matrix obtained with Finite Elements with affine approximation of the electric potential within each triangle of the mesh. On the contrary, the system matrix becomes nonsymmetric for the case of quadrilateral elements; for a typical mesh, the extent of the asymmetry is $\|\mathbf{A} - \mathbf{A}^T\|/\|\mathbf{A}\| \cong 2\%$.

As an application, the analysis of a transmission line has been performed with no free charge $\mathbf{Q}$ according to both the topologies of primal elements; see Fig. 4. The relative permittivity of the dielectric region is 10, while the boundary conditions are $V_e = 0$ V on the external nodes and $V_i = 1$ V on the boundary nodes of the internal conductor.

From the solution of (16) in terms of the nodal potentials vector $\mathbf{V}$, the capacitance of the transmission line has been computed following two approaches: the *charge* approach and the *energy* approach.

A. Charge Approach

The Gauss law (1) can be applied to compute the charge q within a closed surface Σ made of dual faces, surrounding the conductor; then, the capacitance is $C_q = q/(V_i - V_e)$. The

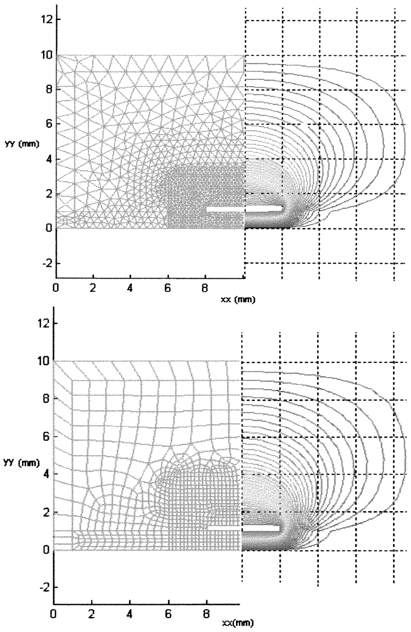

Fig. 4. On the top, the primal mesh based on triangles ($N = 1542$, $L = 4468$, $E = 2926$), and on the bottom, the primal mesh based on quadrilaterals ($N = 1510, L = 2941, E = 1431$), are shown together with the equipotential lines.

Fig. 5. The two closed surfaces surrounding the inner armature of the transmission line are shown for the two mesh topologies.

geometry of Σ is shown in Fig. 5 for both the topologies of elements.

B. Energy Approach

The capacitance is computed from the stored energy as: $C_w = 2W/(V_e - V_i)^2$. In the case of triangles, the stored energy can be expressed as

$$W = \frac{1}{2}\sum_e \mathbf{D}^{eT}\mathbf{E}^e S^e = \frac{1}{8}\sum_e \mathbf{U}^{eT}(\mathbf{A}+\mathbf{B})^T\varepsilon^T(\mathbf{A}+\mathbf{B})\mathbf{U}^e S^e \tag{17}$$

where (6) has been used, together with the constitutive equation $\mathbf{D}^e = \varepsilon^e\mathbf{E}^e$, between uniform fields. S^e is the area of the triangle e. The elemental voltage vector $\mathbf{U}^e$ is computed from the nodal potential of the element e, according to (3), written for the element e. In the case of quadrilaterals, the energy W^e, associated with an element e, is the sum of energies W_i^e associated to each uniformity region R_i, with $i = a, b, c, d$. Using (12) and the constitutive equation between the uniform fields, $\mathbf{E}_i = \varepsilon^{e-1}\mathbf{D}_i$, W^e can be computed as

$$W^e = \frac{1}{2}\sum_i \mathbf{D}_i^{eT}\varepsilon^{e-1}\mathbf{D}_i^e S_i^e = \frac{1}{8}S^e\sum_i \Psi^{eT}\mathbf{A}_i^T\varepsilon^{e-1}\mathbf{A}_i\Psi^e \tag{18}$$

where Ψ^e is related to the nodal potentials of the element e according to (14) and (3), written for the element e. S^e is the area of the quadrilateral element e. Then, the stored energy becomes $W = \sum_e W^e$.

TABLE II
CAPACITANCE VALUES C_q and C_w

Capacitance [F]	*Triangles* *N=1529, E=2926*	*Quadrilaterals* *N=1510, E=1431*
C_q	4.63070 10^{-10}	4.66171 10^{-10}
C_w	4.63070 10^{-10}	4.66179 10^{-10}

The capacitance values C_q and C_w, computed for both the mesh topologies, are reported in Table II. It should be noted that the capacitance value obtained for the case of triangular mesh is coincident with that from Finite Elements with affine behavior of the potential inside the elements. In both the cases of triangles and quadrilaterals, the number of nodal unknowns is almost the same but the number of elements to be processed in the computation of the constitutive equation for the case of triangles is about double that of the corresponding number for the case of quadrilaterals.

V. CONCLUSION

The efficiency of Finite Formulation for electrostatic has been tested on two kinds of primal cell complexes, based on triangular and quadrilateral elements, respectively. The capacitance has been computed according to the *charge* approach and to the *energy* approach; the results are in a good agreement with each other and with FE.

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