3-D Magnetostatic With the Finite Formulation

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***Abstract*—The paper presents the solution of a three-dimensional (3-D) magnetostatic problem using the finite formulation of an electromagnetic field. Two different approaches, based on a primal-dual barycentric discretization of the 3-D space, are presented, considering as unknowns either the magnetic fluxes or the circulations of the vector potential. Results on simple reference configurations are reported and discussed.**

***Index Terms*—Finite formulation, magnetic analysis.**

I. Introduction

THE USE of *finite formulations* of field problems is quite widespread in many subjects of numerical analysis and engineering.

The main concept behind *finite formulations* is related to the use of integral variables, like for instance fluxes and line integrals of field quantities; these variables are then constrained by means of physical laws like balance equations or topological relations between them. As it is put in evidence in the introduction of the Patankar book [1], this approach is a step behind with respect to the differential formulations of field problems. While the differential/variational approach has been fundamental for the development of an analytical treatment of field problems, it can be said that the *finite* approach is more natural for their numerical solution. In many areas of numerical analysis of physical problems the finite volume method has been largely used. In computational fluid dynamics, thermal and multiphysics environments the use of this method is, in fact, very frequent.

In electromagnetic analysis environment the finite approach was related, in the beginning of the use of numerical methods, to finite difference schemes. Following this research line, very important algorithms and computational procedures were developed, for instance the finite difference time domain (FDTD) [2] algorithm as well as the finite integration technique [3]. Other approaches using finite formulations in electromagnetic subjects were naturally tied to circuit expressions of field problems; see, for instance, [4]. Unfortunately, the use of finite difference schemes always have been hindered by their natural application to structured grids. Variational methods, like the finite-element method (FEM), going along well with unstructured grids, have found a much wider application to engineering problems.

The work of Tonti in the definition of *finite formulation of electromagnetic field* (FFEF) [5] has conceptually highlighted the basic aspects of the problems allowing the extensions of the finite approach to generic unstructured space discretizations. One of the main aspects of the work of Tonti is based on the definition of two space grids, which are related by duality topological constraint which fit very well with the structure of electromagnetic equations allowing a straightforward implementation of the theoretical scheme in a computational procedure. This concept is of key importance to an efficient solution of electromagnetic fields as it is well pointed out in [6].

Following this consideration, this work is aimed to the assessment of the use of FFEF in a computational procedure based on an unstructured space discretization. Two different algorithms for the solution of the magnetostatic problem are formulated and implemented in computer codes and their performances tested and compared on simple reference cases.

II. Finite Formulation and Space Discretization

According to the finite formulation, it is possible to deduce a set of algebraic equations directly from physical laws instead of getting them from a discretization process applied to differential or integral equations written in terms of the field quantities. Using *global variables*, like currents or magnetic fluxes, a direct discrete formulation of physical laws can be obtained ready for the numerical implementation. In this work, we focus on the solution of the three-dimensional (3-D) magnetostatic field analysis by reformulating the magnetostatic field laws in a direct discrete way. Global variables are referred to oriented geometrical elements of a system like points $\mathbf{P}$, lines $\mathbf{L}$, surfaces $\mathbf{S}$, and volumes $\mathbf{V}$. These variables are continuous in the presence of different materials and do not require any restriction, like field functions, in terms of derivability conditions on the material media parameters.

The global variables relevant to our magnetostatic problem are reported in Table I. Their dependence on the oriented geometrical elements (in bold face) is evidenced within square brackets; moreover, a tilde is used to specify the outer orientation of the dual entities respect to the inner orientation of the primal ones. *Global variables* are related to the field functions by means of an integration performed on oriented lines, surfaces, volumes, and time intervals; therefore, they are equivalent to the commonly used integral variables.

A further classification of the global variables, in *configuration* and *source* variables, is important in the finite formulation of physical laws. *Configuration variables* describe the pattern of the field while *source variables* describe its sources. The link between configuration and source variables are the *constitutive equations* that contain the material properties and the metrical notions.

A. Cell Complexes

The above classification has a great impact in the numerical applications of the finite formulation. Following the theo-

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TABLE I
GLOBAL VARIABLES OF INTEREST FOR MAGNETOSTATIC AND THEIR RELATION WITH FIELD FUNCTIONS

Configuration global variables (Wb)	Source global variables (A)
magnetic flux: $\Phi[\mathbf{S}] = \int_S \mathbf{B}\cdot d\mathbf{S}$	electric current: $I[\tilde{S}] = \int_{\tilde{S}} \mathbf{J}\cdot d\mathbf{S}$
circulation of the vector potential: $p[L] = \int_L \mathbf{A}\cdot d\mathbf{L}$	magnetic voltage: $F[\tilde{L}] = \int_{\tilde{L}} \mathbf{H}\cdot d\mathbf{L}$

retical scheme, a computational procedure requires the use of a pair of oriented cell complexes $K = \{\mathbf{p}_h, \mathbf{l}_i, \mathbf{s}_j, \mathbf{v}_k\}$ and $\tilde{K} = \{\tilde{\mathbf{v}}_h, \tilde{\mathbf{s}}_i, \tilde{\mathbf{l}}_j, \tilde{\mathbf{p}}_k\}$, one dual of the other, endowed with inner and outer orientation, respectively. Two cell complexes are needed because the configuration variables are referred to a cell complex, the primal, and the source variables are referred to the dual complex. Therefore, the association of the global variables to the respective primal or dual cell complex, both in space and in time, becomes unique. In the paper, the domain of interest is discretized in N_v tetraheda as primal cells $\mathbf{v}_k$, N_s primal faces $\mathbf{s}_j$, N_l primal lines $\mathbf{l}_i$, and N_p primal points $\mathbf{p}_h$.

The dual mesh is built starting from the primal one according to the barycentric subdivision. Inside each tetrahedron $\mathbf{v}_k$, a dual node $\tilde{\mathbf{p}}_k$ is defined in its barycentre. Dual edges $\tilde{\mathbf{l}}_j$ crossing the primal faces $\mathbf{s}_j$ are built in two segments, each one going from a dual node to the barycentre of the face. These edges inherit their outer orientation by the inner orientation of the primal face. A dual face $\tilde{\mathbf{s}}_i$ is defined by the primal edge crossing it; also in this case, the orientation of the dual face is induced by the one of the corresponding primal edge. Inside a tetrahedron, four portions of four dual faces are tailored; see Fig. 1. Each of these portions of dual faces is a quadrilateral, defined by the midpoint of a primal edge, by two faces barycentres, and by the dual node.

Due to the duality between the two cell complexes K and $\tilde{K}$, it can be stated that N_v is the number of $\tilde{\mathbf{p}}_k$, N_s is the number of $\tilde{\mathbf{l}}_j$, N_l is the number of $\tilde{\mathbf{s}}_i$, and N_p is the number of $\tilde{\mathbf{v}}_h$.

B. Physical Laws in Finite Form

In *finite form*, physical laws of electromagnetism are expressed as topological constraints on variables of the same kind. These constraints tie a global variable associated to a geometrical element to be equal to another global variable associated to its boundary. In the case of magnetostatic, with reference to a pair of primal dual cell complexes K-$\tilde{K}$ above defined, the *magnetic Gauss law* can be written as

$$[\mathbf{D}]\{\Phi\} = 0 \tag{1}$$

where $[\mathbf{D}]$ is the incidence matrix between a primal cell $\mathbf{v}_k$ and a primal face $\mathbf{s}_j$, of dimension $N_v \times N_s$ and Φ the vector of fluxes, the *Ampère law* becomes

$$[\tilde{\mathbf{C}}]\{\mathbf{F}\} = \{\mathbf{I}\} \tag{2}$$

where $[\tilde{\mathbf{C}}]$ is the incidence matrix between a dual face $\tilde{\mathbf{s}}_i$ and a dual line $\tilde{\mathbf{l}}_j$, of dimension $N_l \times N_s$. $\{F\}$ is the vector of magnetic

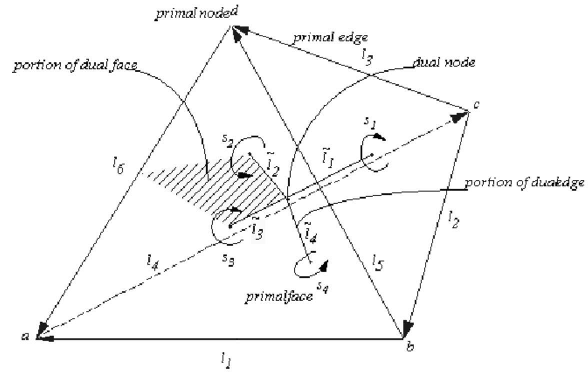

Fig. 1. A generic primal cell (a tetrahedron) is shown with the corresponding primal faces, edges, and nodes. Correspondingly, the dual node, the portions of dual edges, and dual faces inside the primal cell are shown. Local numbering is displayed together with inner and outer orientations of some geometrical elements.

voltages and $\{I\}$ is the vector of currents. Moreover, from the conservation of charge, it can be written as

$$[\tilde{\mathbf{D}}]\{\mathbf{I}\} = 0 \tag{3}$$

where $[\tilde{\mathbf{D}}]$ is the incidence matrix between a dual volume $\tilde{\mathbf{v}}_h$ and a dual face $\tilde{\mathbf{s}}_i$ of dimension $N_p \times N_l$.

III. CONSTITUTIVE EQUATION

The link between configuration and source variables is the *constitutive equations* that contain material properties and metrical notions [7]. In the case of magnetostatic and with reference to a tetrahedron $\mathbf{v}_k$, of vertices *a, b, c, d* (Fig. 1), the local vector of fluxes can be introduced as $\Phi^k = [\Phi_1 \ldots \Phi_4]^T$ together with the local vector of magnetic voltages $\mathbf{F}^k = [F_1 \ldots F_4]^T$ relative to the portions of dual edges inside $\mathbf{v}_k$. Under the hypothesis of uniformity of the fields $\mathbf{B}^k$ and $\mathbf{H}^k$ and homogeneity of the media permeability μ^k inside the tetrahedron, the $\mathbf{H}^k$ vector can be expressed as

$$\mathbf{H}^k = \frac{1}{\mu^k}\sum_{j=1}^{4} d_{kj}\Phi_j \mathbf{w}_j \tag{4}$$

where $\mathbf{w}_j$ are the usual facet element shape vectors defined on the basis of the scalar affine nodal functions N_i associated to the ith node of the tetrahedron; d_{kj} are the incidence numbers between the inner orientations of the primal cell $\mathbf{v}_k$ and of its primal faces $\mathbf{s}_j$. Then, from (4), the vector of the magnetic voltages $\mathbf{F}^k$ can be written as

$$\{\mathbf{F}^k\} = \left[\tilde{\mathbf{L}}\right]\{\mathbf{H}^k\} \tag{5}$$

where $[\tilde{\mathbf{L}}]$ is the 4×3 matrix whose rows are the vectors relative to the four portions of dual edges internal to $\mathbf{v}_k$. Combining (5) and (4), the local constitutive equation can be written as

$$\{\mathbf{F}^k\} = \lfloor \mathbf{M}^k \rfloor \{\Phi^k\} \tag{6}$$

where $\lfloor \mathbf{M}^k \rfloor$ is the square nonsymmetric elemental matrix of dimension four relative to the kth cell $\mathbf{v}_k$. From (6), the global constitutive equation can be assembled working element by element

$$\{\mathbf{F}\} = \lfloor \mathbf{M} \rfloor \{\Phi\} \tag{7}$$

where $[\mathbf{M}]$ is the global constitutive matrix of dimension N_s, still nonsymmetric.

IV. Magnetostatic Formulations

Two approaches can be followed in the finite formulation of magnetostatic, assuming as known the external currents and the permeability distribution. The first based on the fluxes Φ and the second based on the circulations of the vector potential p as unknowns.

A. Φ Formulation

In this case, N_s fluxes relative to the primal faces are considered; N_{se} of them are assigned, as boundary conditions on the external faces. Boundary conditions must comply with (1), thus the assigned flux must be null. N_{si} fluxes are the real unknowns relative to the internal faces, with $N_s = N_{se} + N_{si}$.

As concerns the equations, the magnetic Gauss law (1) provides $N_v - 1$ independent equations because one constraint is linearly dependent by the assigned boundary condition. In order to constraint the N_{si} unknown fluxes $N_{si} - N_v + 1$, Ampère law (2) must be imposed. Following a scheme similar to the one used in network analysis, a tree can be defined along dual edges and Ampere law can be imposed on the fundamental loops associated to the branches of the *cotree*. Using the constitutive equation (7) equations can be written as

$$[\tilde{\mathbf{C}}][\mathbf{M}]\{\Phi\} = \{\mathbf{I}\}. \tag{8}$$

B. p Formulation

The N_l circulations of the vector potential relative to primal edges can be introduced such that

$$[\mathbf{C}]\{\mathbf{p}\} = \{\Phi\} \tag{9}$$

where $[\mathbf{C}]$ is the incidence matrix between a primal face $\mathbf{s}_j$ and a primal line $\mathbf{l}_i$, of dimension $N_s \times N_l$; $\{p\}$ is the vector of circulations of the vector potential. Because of the identity $[D][\mathbf{C}] = 0$, (9) identically satisfies the Gauss law (1). Substituting (9) into (8), the following N_l equations can be derived:

$$[\tilde{\mathbf{C}}][\mathbf{M}][\mathbf{C}]\{\mathbf{p}\} = \{\mathbf{I}\} \tag{10}$$

of which only $N_l - N_p + 1$ are independent due to (3). This means that $N_p - 1$ circulations can be eliminated in the p vector, associated to the branches of a tree based on N_l primal lines and N_p primal nodes. This is analogous to the Albanese–Rubinacci gauge condition on the vector potential [8]. It can be shown, as in [5], that $[\tilde{\mathbf{C}}] = [\mathbf{C}]^T$ and that the resulting matrix of the system (10) $[\mathbf{C}]^T[\mathbf{M}][\mathbf{C}]$ is symmetric even if $[\mathbf{M}]$ is not, as shown in [9].

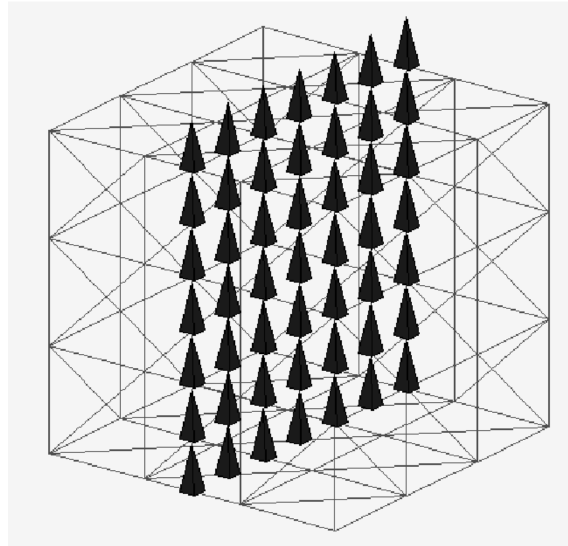

Fig. 2. Cubic domain with a uniform flux. Vectors of the computed magnetic flux density.

C. Comparison of Formulations

If the flux formulation has a direct link with physics, the resulting system obtained from (1) and (8) leads to a nonsymmetric matrix. Moreover, the computational scheme requires us to store for each dual cotree branch the fundamental loop topology. This fact, in large 3-D problems, can be very demanding in terms of memory requirements. In addition, with usual tetrahedral mesh generators the obtained algebraic system is larger for the Φ formulation respect to the p one. A further advantage of the p formulation is that it can be easily extended to the case of magneto-quasistatic problems.

V. Numerical Experiments and Discussion

The proposed numerical schemes have been implemented in a numerical analysis environment allowing us to define the required geometrical entities and their topological relationships. Some simple analysis problems have been tested in order to assess the accuracy of the methods.

A. Cube With Uniform Field

This problem has been solved to evaluate the accuracy of the numerical schemes on a simple case discretized by means of an irregular primal mesh made of simplexes. The domain of the problem is a cube with a flux entering the bottom face and exiting the upper one with an average magnetic flux density value of 1 T; see Fig. 2. Both schemes could represent correctly the uniform field inside the cube with a relative accuracy better than 10^{-11}.

B. Ferromagnetic Sphere in a Uniform Field

This problem has a analytical solution which can be used as reference. The domain of the problem is shown in Fig. 3, together with the simplex primal mesh. The primal mesh is made of 3240 tetrahedra, 6676 faces, 4088 edges, and 653 nodes. A uniform flux is imposed on the two bases of the external cylinder

TABLE II
FERROMAGNETIC SPHERE IN UNIFORM FIELD COMPARISONS WITH ANALYTICAL SOLUTION

	Φ formulation	p formulation
B in the center of the sphere [T]	2.853	2.894
Error wrt analytical solution [%]	4.6	3.2

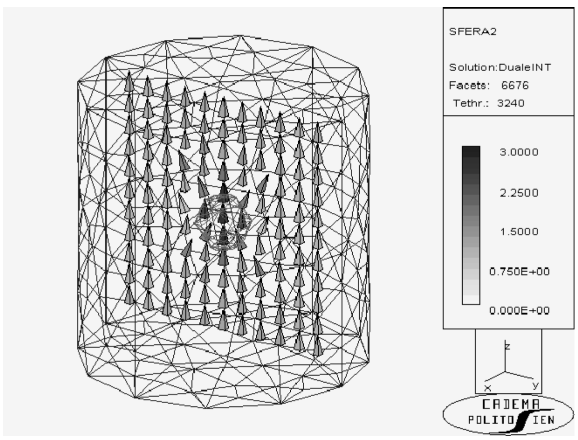

Fig. 3. Ferromagnetic sphere immersed in a uniform field. Mesh used for the numerical solution and vector plot of magnetic flux density.

correspondent to a uniform magnetic flux density of 1 T, while the sphere has a value of relative permeabilty of 1000.

The analytical solution for the magnetic flux density inside the sphere is given by

$$B = \frac{3\,(\mu_r - 1)}{\mu_r + 2} B_0 = 2.991 \text{ T}. \tag{11}$$

The results obtained by the two schemes are reported in Table II, while a sketch of the tetrahedral mesh together with a vector plot of the magnetic flux density can be seen in Fig. 3.

The error with respect to the analytical solution can be attributed both to a discretization error of the spherical surface and to a uniform flux condition imposed at a finite distance from the sphere and not to infinity.

VI. CONCLUSION

The work performed has allowed us to test the efficiency of the numerical procedures based on the FFEF applied to unstructured meshes. The two proposed formulations have given good results when compared to simple reference cases, while some important issues related to numerical and memory requirements seem to give a preference to the p formulation. The work will continue studying the time-varying case.

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