Finite Formulation of Electromagnetic Field

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Abstract—**We show that the equations of electromagnetism can be directly obtained in a finite form, i.e., discrete, thus avoiding the traditional discretization methods of Maxwell's differential equations. The finite formulation can be used with unstructured meshes in two and three dimensions and easily permits to obtain fourth-order convergence.**

Index Terms—**Computational electromagnetism, convergence of numerical methods, discrete electromagnetism, finite formulation.**

I. INTRODUCTION

COMPUTATIONAL electromagnetism is commonly based on a discretization of Maxwell's field equations. We show that it is possible to express the laws of electromagnetism starting directly from experimental facts by a set of algebraic equations [1]–[3].

The finite formulation is based on five items:

1) the systematic use of *global variables* instead of field functions;
2) the distinction between *source, configuration* and energy *variables*;
3) the use of *space-time elements* endowed with *inner* and *outer* orientations;
4) the use of *two cell complexes,* a primal one endowed with inner orientation and a dual one endowed with outer orientation;
5) the fact that *global variables are related to oriented space–time elements.*

A. Global Variables

We use the term *global* variable as synonym of *integral* variable. Physical measurements deal mainly with *global variables*, such as voltages, fluxes, charge contents, and charge flows, not field vectors. Global variables are continuous through the separation surface of two materials while field variables are not. This makes global variables best suited to deal with regions made of different materials.

While field variables are indispensable in a differential formulation, global variables are the natural tool for a finite formulation. Contrary to field functions, which are functions of points and instants, global variables are *domain functions* and the space and time elements to which they are related will be put inside square brackets.

The time integral of a physical variable, say, E, will be called its *impulse* and will be denoted by the corresponding calligraphic letter, say, $\mathcal{E}$.

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TABLE I
FIELD AND GLOBAL VARIABLES OF ELECTROMAGNETISM

	source				*config.*	
field	ρ	$\mathbf{J}$	$\mathbf{D}$	$\mathbf{H}$	$\mathbf{E}$	$\mathbf{B}$
space global	Q^c	I	Ψ	F	E	Φ
space-time global	Q^c	Q^f	Ψ	$\mathcal{F}$	$\mathcal{E}$	Φ

It is expedient to distinguish between global variables *in space* and global variables *in space–time.* So, electric current I, electromotive force E, and magnetomotive force F are global variables in space while electric charge flow Q^f, electromotive force impulse $\mathcal{E}$, and magnetomotive force impulse $\mathcal{F}$ are global variables in space–time.

B. Configuration, Source and Energy Variables

Source variables are, first of all, those that describe the source of the electromagnetic field, i.e., charges and currents and, in the second place, all variables linked to them by algebraic or differential operations without the intervention of physical constants. Table I collects the six main variables of this kind. *Configuration variables* are those that describe the configuration of the field, its potentials and all those variables that are linked to them by algebraic or differential operations without the intervention of physical constants. They are linked to the source ones by the constitutive equations. *Energy variables* are those obtained by the product of one source variable and one configuration variable: examples are the electric energy density $w_E = \mathbf{D} \cdot \mathbf{E}/2$, the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.

C. Oriented Space–Time Elements

With this name we mean points ($\mathbf{P}$), lines ($\mathbf{L}$), surfaces ($\mathbf{S}$), volumes ($\mathbf{V}$), time instants ($\mathbf{I}$), and time intervals ($\mathbf{T}$). A further property of space and time elements, not commonly stressed, is that these elements can be endowed with two kinds of orientations, as shown in Fig. 1: the *inner* and the *outer* one. The four space elements endowed with inner orientation will be denoted by $\mathbf{P}$, $\mathbf{L}$, $\mathbf{S}$, $\mathbf{V}$, while those endowed with outer orientation will be denoted by $\tilde{\mathbf{P}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{S}}$, $\tilde{\mathbf{V}}$. In an analogous way, the primal and dual time elements will be denoted by $\mathbf{I}$, $\mathbf{T}$ and $\tilde{\mathbf{I}}$, $\tilde{\mathbf{T}}$, respectively, as shown in Fig. 3.

D. Cell Complexes

Since a finite formulation requires space and time elements, not only points and instants, it appear natural to introduce *cell complexes* instead of coordinate systems. Cell complexes exhibit vertices, edges, faces, and volumes. In a finite formulation, a pivotal role is played by the *dual* complex. If we make use of a simplicial complex as a primal complex, then the commonest