Convergence and Stability of the Cell Method with Non Symmetric Constitutive Matrices

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***Abstract*—Starting from experimental facts, Cell Method allows to reformulate electrodynamic laws in a finite form. In the present paper we present some constitutive relations that do not lead to symmetric and positive definite constitutive matrices. Despite of this, Cell Method is stable and has got a high order of convergence.**

I. Introduction

Cell Method is a numerical methodology born from a deep analysis of the geometrical structures common to many physical theories [1]. The result is a set of topological equations, formulated on two cell complexes, linking some global variables as fluxes, impulse voltages and charges [2]. We can reach this goal directly from Maxwell's equations:

- By differential forms [3][4]
- By integral formulation [5]

By means of correct assignments of global physical variables to geometrical elements of two space-time cell complexes and through a second order approximation, we can express electrodynamic laws as follows:

On primal cell complex

$$\begin{gathered} CV^{n-1/2} = -(\Phi^n - \Phi^{n-1})/\Delta t \\ D\Phi^n = 0 \end{gathered} \tag{1}$$

On dual cell complex

$$\begin{gathered} \tilde{C}F^n = (\Psi^{n+1/2} - \Psi^{n-1/2})/\Delta t + I^n \\ \tilde{D}\Psi^{n-1/2} = {Q_c}^{n-1/2} \end{gathered} \tag{2}$$

where $V^{n-1/2}$, Φ^n, F^n, $\Psi^{n-1/2}$, I^n, ${Q_c}^{n-1/2}$ are some scalar array while G, C, D and $\tilde{G} = D^T$, $\tilde{C} = C^T$, $\tilde{D} = -G^T$ are the incidence matrices related to the primal and dual cell complexes respectively.

II. Constitutive relations

The constitutive relations can be expressed as follows:

$$\Phi^n = M_\mu F^n \tag{3}$$

$$\Psi^{n-1/2} = M_\varepsilon V^{n-1/2} \tag{4}$$

$$I^n = M_\sigma (V^{n+1/2} + V^{n-1/2})/2 \tag{5}$$

If an orthogonality occurs (primal edges $\perp$ dual faces and primal faces $\perp$ dual edges) the constitutive matrices have a simple structure. For example, for the magnetic constitutive relation, we can think that

$$\Phi_\beta{}^n = \mu(s_\beta/\tilde{l}\beta)F_\beta{}^n \tag{6}$$

between variables matched to geometrical elements in a one-to-one mapping (like s_β and $\tilde{l}\beta$). If these elements have a compatible orientation, then the constitutive matrices are symmetric (diagonal) and positive definite.

When the orthogonality does not occur and we use dual barycentric grids, we can obtain some constitutive relations in another way. Let us take a primal grid, made by cells c_i with an arbitrary number of faces n_i and let us build the dual one. The superposition of the primal and the dual grid leads to a subdivision of the domain by quadrilateral (2D) or hexahedrical (3D) cells that we will call *microcells* [7]. Let us assume a uniform field in each of them. Now, for each cell c_i with n_i faces, a primal one for example, we can obtain a constitutive matrix M_μ^i linking n_i magnetic flux values with n_i magnetic voltage values by means of a composition of **B** and **H** fields of the microcells forming the primal cell (Fig.1). Then, building the M_μ^i of each cell, we can get to the global constitutive matrix M_μ in (3).

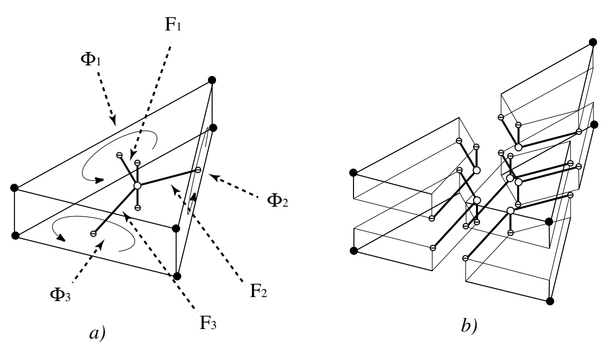

Fig. 1. Primal cell (a) and its subdivision in microcells (b)

What about these matrices?

1. M_μ^i and M_μ matrices were always invertible, in all the cases we have analyzed.

2. M_μ matrices are not symmetric in general cases. They are symmetric (and positive definite) when an orthogonality occurs between primal and dual grids.

We can do similar considerations for M_ε and M_σ. Let us see how they work.

III. Electrostatic

In finite terms, Laplace's equation without boundary conditions is: $G^T M_\varepsilon G = 0$, where G is the primal cell complex edge-vertex incidence matrix. We have solved this equation on a 2D square domain [0,1]x[0,1], with potential assignments on the boundary, using first primal triangular then primal quadrilateral cells, with dual barycentric grids, and having homogeneous or non homogeneous media inside. In all examined cases we have seen that:

- For primal triangular cells, $-G^T M_\varepsilon G$ and $-G^T (M_\varepsilon)^{-1} G$ matrices were always *symmetric and positive definite*
- For primal quadrilateral cells, $-G^T M_\varepsilon G$ and $-G^T (M_\varepsilon)^{-1} G$ matrices were not symmetric, but their eigenvalues were *real and positive* (some eigenvalues may not be real, but it seems so because of rounding errors: in fact imaginary parts are usually very small compared to the real parts ($< 0.5\%$); moreover this percentage decreases for regular grids, while it increases for very irregular ones).

In order to value the convergence order, we have used some harmonic test functions as $\varphi(x, y) = exp(x)sin(y)$ with structured (s) and non structured (ns) grids. We have analyzed also the convergence order we can obtain using Romberg's method. In this case we have used two primal cell complexes, the first one with sizes thicker twice in comparison to the second one. The results are displayed in Table 1.

IV. Electrodynamics

From the electrodynamic laws (1)(2) we can obtain the Helmholtz's equations in frequency domain:

$$\begin{array}{c} C(M_\varepsilon)^{-1} C^T F = \omega^2 M_\mu F \\ C^T (M_\mu)^{-1} C V = \omega^2 M_\varepsilon V \end{array} \quad (7)$$

In order to avoid spurious solutions, involving instabilities in time domain, a sufficient condition is that M_μ, M_ε matrices are *symmetric and positive definite* [3][6]. A weaker condition is that $C(M_\varepsilon)^{-1}C^T$ has *real and positive eigenvalues* and M_ε is *symmetric and positive definite*. It is possible to have such a condition in 2D cases of TM (or TE) modes calculus for resonant cavities. In fact, in a 2D case of TM mode calculus, where $\mathbf{H} = Hx\mathbf{i} + Hy\mathbf{j}$, $\mathbf{E} = Ez\mathbf{k}$, the matrices C, C^T becomes G,$-G^T$ respectively, so an Helmholz's equations becomes:

$$-G^T (M_\mu)^{-1} G V = \omega^2 M_\varepsilon V \quad (8)$$

TABLE I

Cell kind	Convergence order			
	microcells		micr.+Romb.	
	s	ns	s	ns
triangles	2	$\simeq 2$	4	$\simeq 3.5$
quadr.	2	$\simeq 2$	4	$\simeq 3.4$

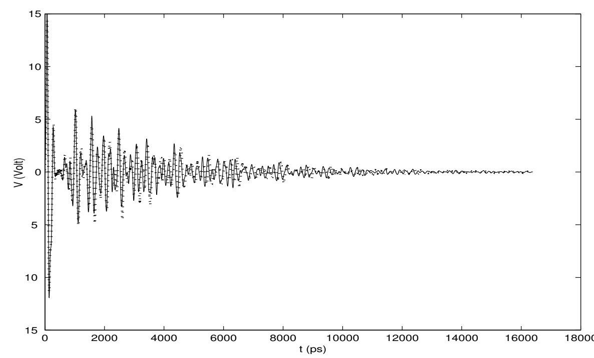

Fig. 2. Input voltage of a resonant cavity excited by a gaussian current impulse

Building M_μ with the microcells and being M_ε diagonal and positive definite, equation (8) will have only *real and positive eigenvalues* without spurious modes. Tests made on over 16000 time steps confirm the absence of instabilities (Fig.2) [7].

V. Conclusion

New general constitutive relations, built without using particular shape functions and without symmetry requirements, has been proposed. They lead to high convergence order and stability in 2D time domain problems.

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