A Novel Approach to Deriving a Stable Hybridized FDTD Algorithm Using the Cell Method

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1 Introduction

In many practical situations it becomes necessary to combine two algorithms, e.g., the FDTD and FETD to improve the accuracy of the solution without placing an inordinately heavy burden on the cpu. For instance, it is often desirable to use the FDTD on a coarse, structured Cartesian mesh bulk of the region and to employ either a FDTD subgridding scheme in a smaller region containing objects with fine features, or to use the FETD on an unstructured mesh to further improve the modeling accuracy in this region. Although this circumvents the problem of having to use a very small time step throughout the computational domain to satisfy the Courant condition [1] which is associated with the smallest length of the mesh edges in the entire computational domain, it does introduce the burden of temporal and spatial interpolations and the distinct possibility of instabilities introduced by the spurious reflections at the interface of the two domains with dissimilar meshes. What is even more disconcerting is that although the two algorithms, applied in the different domains, may be independently stable (in the absence of the other domain), the combination may still be unstable.

In this paper we propose two new algorithms, developed by using the recently-introduced Cell Method (CM [2],[3]) that enables us to address both the problems of instability and connectivity when dealing with a combination of a coarse mesh FDTD in one domain, and either a fine mesh or a triangular one in the other.

2 The Cell Method

The Cell Method is a numerical procedure that enables us to formulate the laws of electromagnetics in a compact algebraic form. The CM utilizes the global (integral) variables (electric voltage impulse $\mathcal{V}[\mathbf{L},\mathbf{T}](=\int_T\int_L \mathbf{E}\cdot d\mathbf{l}dt)$, electric flux $\Psi[\tilde{\mathbf{S}},\tilde{\mathbf{I}}](=\int_{\tilde{S}} \mathbf{D}\cdot d\mathbf{s})$, magnetic voltage impulse $\mathcal{F}[\tilde{\mathbf{L}},\tilde{\mathbf{T}}](=\int_{\tilde{T}}\int_{\tilde{L}} \mathbf{H}\cdot d\mathbf{l}dt)$, magnetic flux $\Phi[\mathbf{S},\mathbf{I}](=\int_S \mathbf{B}\cdot d\mathbf{s})$ and electric current flow $Q_f[\tilde{\mathbf{S}},\tilde{\mathbf{T}}](=\int_{\tilde{T}}\int_{\tilde{S}} \mathbf{J}\cdot d\mathbf{s}dt))$ as representations of the physical variables in contrast to the local ones such as the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$. Also, it utilizes two staggered grids *viz.*, primal and dual, whose oriented elements (primal points $\mathbf{P}$, lines $\mathbf{L}$, surfaces $\mathbf{S}$, volumes $\mathbf{V}$ and dual points $\tilde{\mathbf{P}}$, lines $\tilde{\mathbf{L}}$, surfaces $\tilde{\mathbf{S}}$, volumes $\tilde{\mathbf{V}}$) are associated with the global variables - for instance the primal line $\mathbf{L}$ with the electric voltage impulse $\mathcal{V}$ and the primal surface $\mathbf{S}$ with the magnetic flux Φ. In CM the electrodynamic laws can be divided into two categories, based on the type of variables that are linked. The categories are:

Topological equations (field equations). They link the physical variables with the geometrical elements belonging to the same grid (the primal or the dual one). In the 2D $\mathrm{TM_z}$ case

the equations reduce to:
$\mathbf{\Phi}^{n+1/2} = \mathbf{\Phi}^{n-1/2} - \mathbf{G}\boldsymbol{\mathcal{V}}^n, \qquad \mathbf{\Psi}^{n+1} = \mathbf{\Psi}^n + \mathbf{G}^T\boldsymbol{\mathcal{F}}^{n+1/2} - \mathbf{Q}_f^{n+1/2}$
where $\boldsymbol{\mathcal{V}}^n$, $\mathbf{\Phi}^{n+1/2}$, $\boldsymbol{\mathcal{F}}^{n+1/2}$, $\mathbf{\Psi}^n$, $\mathbf{Q}_f^{n+1/2}$ are arrays of scalars and $\mathbf{G}$ is the incidence matrix associated with the primal points and lines.
Metric relations. These relations link the global variables associated with geometrical elements belonging to different type of grids. The metric relations can be split into two parts via the introduction of the intermediate variables *viz.*, the electric voltage $V[\mathbf{L}, t)(= \int_L \mathbf{E}\cdot d\mathbf{l})$, the magnetic voltage $F[\tilde{\mathbf{L}}, t)(= \int_{\tilde{L}} \mathbf{H}\cdot d\mathbf{l})$ and the electric current $I[\tilde{\mathbf{S}}, t)(= \int_{\tilde{S}} \mathbf{J}\cdot d\mathbf{s})$. Next we turn to the constitutive and time integration relations that are given below:
Constitutive relations: These relations are expressed as:

$$\mathbf{V}^n = \mathbf{M}_{\varepsilon i}\mathbf{\Psi}^n\ , \mathbf{F}^{n+1/2} = \mathbf{M}_{\nu}\mathbf{\Phi}^{n+1/2} \text{ or } \mathbf{\Psi}^n = \mathbf{M}_{\varepsilon}\mathbf{V}^n\ , \mathbf{\Phi}^{n+1/2} = \mathbf{M}_{\mu}\mathbf{F}^{n+1/2} \quad (1)$$

where the matrices $\mathbf{M}_{\varepsilon i}$, $\mathbf{M}_{\nu}$, $\mathbf{M}_{\varepsilon}$, $\mathbf{M}_{\mu}$ are called *constitutive matrices*. The microcell interpolation scheme (MIS)[2] may be employed to build them.
Time integration relations: In this work we explore two different types of time integration schemes. Given the time step τ, the first is chosen to be:

$$\boldsymbol{\mathcal{V}}^n = \tau\mathbf{V}^n, \quad \boldsymbol{\mathcal{F}}^{n+1/2} = \tau\mathbf{F}^{n+1/2}, \quad \mathbf{Q}_f^{n+1/2} = \tau\mathbf{I}^{n+1/2} \quad (2)$$

and its use leads to the *leap-frog scheme* used in the conventional FDTD [1]. The second one reads:

$$\boldsymbol{\mathcal{V}}^n = \frac{\tau}{4}(\mathbf{V}^{n+1} + 2\mathbf{V}^n + \mathbf{V}^{n-1}), \quad \boldsymbol{\mathcal{F}}^{n+1/2} = \tau\mathbf{F}^{n+1/2}, \quad \mathbf{Q}_f^{n+1/2} = \tau\mathbf{I}^{n+1/2} \quad (3)$$

and leads to the *Newmark time-stepping scheme* utilized in FETD.

3 Formulation of hybrid algorithms

In order to set up the hybridization it is necessary to partition the computational domain Ω into two subdomains. The first one of these, say subdomain Ω_1, utilizes a coarse rectangular mesh on which the original FDTD algorithm is employed. On the second subdomain Ω_2, which has the fine details, we choose to use a different form of algorithm whose maximum time step is independent of the minimum mesh size in Ω_2. The partition of the domain Ω leads to a partition of the incidence matrix $\mathbf{G}$, and the constitutive matrices $\mathbf{M}_{\varepsilon i}$, $\mathbf{M}_{\nu}$ into four blocks. We will use the matrices to generate the time domain algorithms separately for the two domains. For the first domain Ω_1, the topological equations and the metric relations to derive the CM/FDTD algorithm take the form:
1)$\mathbf{F}_1^{n+1/2} = \mathbf{F}_1^{n-1/2} - \tau\mathbf{M}_{\nu 11}(\mathbf{G}_{11}\mathbf{V}_1^n + \mathbf{G}_{12}\mathbf{V}_2^n)$
2)$\mathbf{V}_1^{n+1} = \mathbf{V}_1^n + \tau\mathbf{M}_{\varepsilon i 11}(\mathbf{G}_{11}^T\mathbf{F}_1^{n+1/2} - \mathbf{I}_1^{n+1/2})$
For the domain Ω_2 we offer two options, *viz.*, the CM/NEW and the CM/FDTD algorithm. The former is based on the (3) and, hence, is an implicit algorithm. It can be expressed as:
$(\mathbf{M}_{\varepsilon 22} + (\tau^2/4)\mathbf{G}_{22}^T\mathbf{M}_{\nu 22}\mathbf{G}_{22})\mathbf{V}_2^{n+1} = (2\mathbf{M}_{\varepsilon 22} - (\tau^2/2)\mathbf{G}_{22}^T\mathbf{M}_{\nu 22}\mathbf{G}_{22} - \tau^2\mathbf{G}_{12}^T\mathbf{M}_{\nu 11}\mathbf{G}_{12})\mathbf{V}_2^n - (\mathbf{M}_{\varepsilon 22} + (\tau^2/4)\mathbf{G}_{22}^T\mathbf{M}_{\nu 22}\mathbf{G}_{22})\mathbf{V}_2^{n-1} - \tau^2\mathbf{G}_{12}^T\mathbf{M}_{\nu 11}\mathbf{G}_{11}\mathbf{V}_1^n - \tau(I_2^{n+1/2} - I_2^{n-1/2})$.
In contrast to CM/NEW, the CM/FDTD algorithm uses (2), with time subgridding, and is an explicit algorithm. It utilizes the update equations:
1)$\mathbf{F}_2^{n+1/6} = \mathbf{F}_2^{n-1/6} - (\tau/3)\mathbf{M}_{\nu 22}\mathbf{G}_{22}\mathbf{V}_2^n$

2)$\mathbf{V}_2^{n+1/3} = \mathbf{V}_2^n + (\tau/3)\mathbf{M}_{\varepsilon i22}(\mathbf{G}_{12}^T\mathbf{F}_1^{n+1/2} + \mathbf{G}_{22}^T\mathbf{F}_2^{n+1/6} - \mathbf{I}_2^{n+1/6})$
3)$\mathbf{F}_2^{n+3/6} = \mathbf{F}_2^{n+1/6} - (\tau/3)\mathbf{M}_{\nu 22}\mathbf{G}_{22}\mathbf{V}_2^{n+1/3}$
4)$\mathbf{V}_2^{n+2/3} = \mathbf{V}_2^{n+1/3} + (\tau/3)\mathbf{M}_{\varepsilon i22}(\mathbf{G}_{12}^T\mathbf{F}_1^{n+1/2} + \mathbf{G}_{22}^T\mathbf{F}_2^{n+3/6} - \mathbf{I}_2^{n+3/6})$
5)$\mathbf{F}_2^{n+5/6} = \mathbf{F}_2^{n+3/6} - (\tau/3)\mathbf{M}_{\nu 22}\mathbf{G}_{22}\mathbf{V}_2^{n+2/3}$
6)$\mathbf{V}_2^{n+1} = \mathbf{V}_2^{n+2/3} + (\tau/3)\mathbf{M}_{\varepsilon i22}(\mathbf{G}_{12}^T\mathbf{F}_1^{n+1/2} + \mathbf{G}_{22}^T\mathbf{F}_2^{n+5/6} - \mathbf{I}_2^{n+5/6})$

Note that these algorithms can be used with a triangular mesh as well. We have verified that with a subgridding scheme in the region Ω_2 the CM/NEW algorithm is stable for $\tau < \frac{1}{c_0\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}}}$ whereas the CM/FDTD algoritm is stable for $\tau < \frac{0.95}{c_0\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}}}$ where Δx and Δy are the grid step sizes in the domain Ω_1.

4 Numerical results

For the first example, we consider a $\mathrm{TM_Z}$ wave propagating in an empty parallel-plate waveguide, with a view to evaluating the level of reflection at the interface between the domains Ω_1 and Ω_2. The reflection coefficient $R = |V_{CM}(\omega) - V_{FDTD}(\omega)|/|V_{FDTD}(\omega)|$ has been calculated in the frequency range of 0.8-1.5 GHz, by using the two algorithms (Fig.3a) in Ω_2 that also utilize two different meshes (Fig.2). In the second test, we compare the numerical results for the longitudinal field $\mathrm{E_z}$, generated by a line source in the domain Ω_2 that contains the fine mesh (subgridding and triangular mesh). The relative error $|E_{zCM}(\omega) - E_{zEXACT}(\omega)|/|E_{zCM}(\omega)|$, calculated at the frequency for which $\lambda = 20\Delta x$, is shown in Fig 3b.

5 Conclusions

In this paper two new hybrid FDTD algorithms, formulated via the Cell Method, have been presented for the analysis of structures with fine details. The level of reflection introduced by the interface of dissimilar mesh (subgridding or triangular mesh) in two different regions of the computational domain has been studied, and the results are presented for different approaches for the sake of comparison. The paper also shows that the accuracy of the field computation can be significantly improved by using a fine triangular mesh and the CM/NEW algorithm in which the time step used in the same in the two domains.

References

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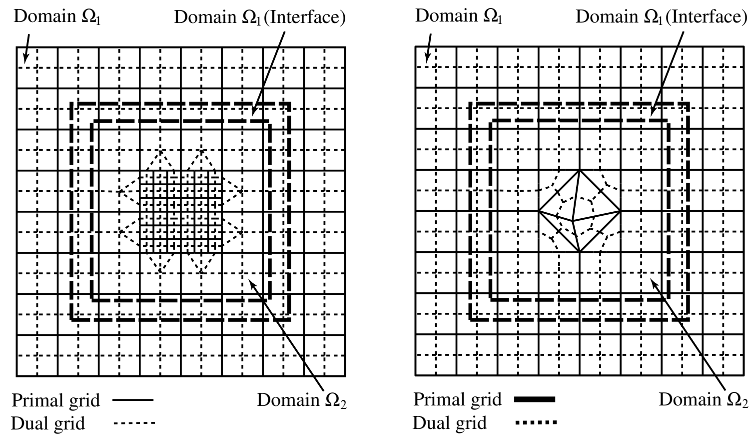
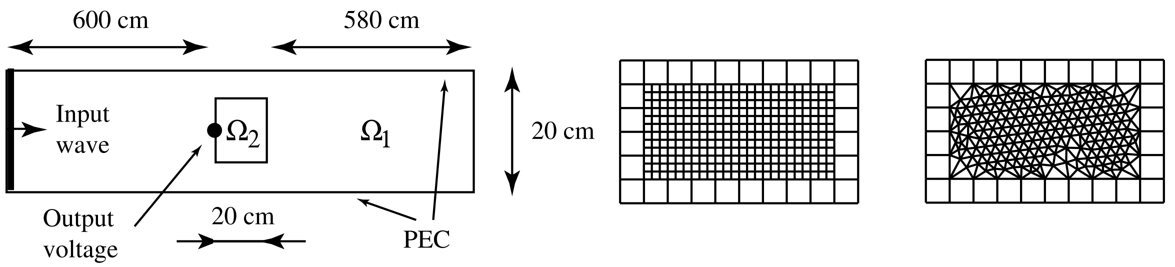


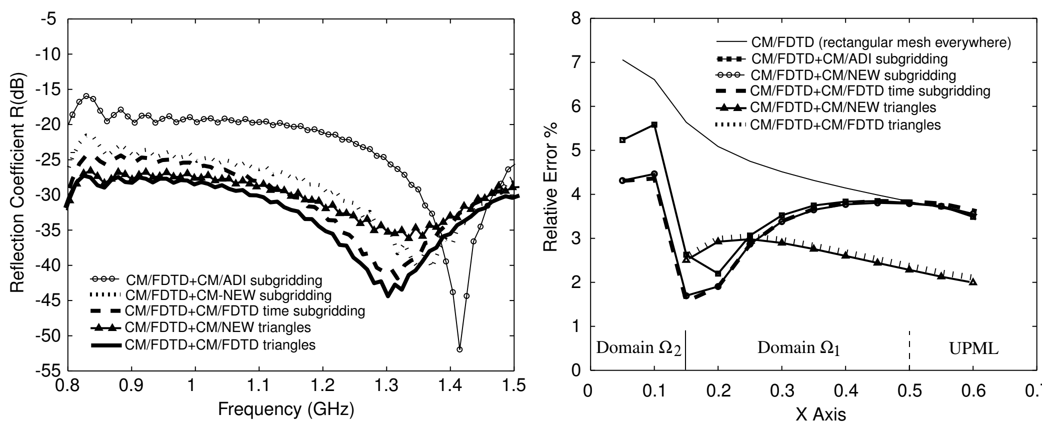
a) Square grid+subgridding b) Square grid+triangular grid

Figure 1: Example of two subdomains. The domain Ω_1 with a square mesh and the domain Ω_2 with (a) a square mesh with subgridding or (b) a square mesh with a triangular mesh.



a) Parallel plate waveguide b) Subgridding domain Ω_2 c) Triangular domain Ω_2

Figure 2: (a) Waveguide for the first application (b) Subgridding in the domain Ω_2 (c) Triangular mesh in the domain Ω_2



a) b)

Figure 3: (a) Reflection coefficient R at the interface between the domains Ω_1 and Ω_2; (b) Relative error in the radial electric field E_z generated by a line source.